1. Given the following collections of sets, find in each of parts (a), (b), and (c) all relationships of equality, subset, and proper subset existing between pairs of them.

(a) \( A = \{-1, 1, 4\}, B = (-1, 4), C = \{x \in \mathbb{R} \mid x^3 - 4x^2 - x + 4 = 0\}, D = [-1, 4] \)

   Solution: Note that since the polynomial \( x^3 - 4x^2 - x + 4 \) is equal to \((x + 1)(x - 1)(x - 4)\), the sets \( A \) and \( C \) are equal. \( B \subset D \) and \( A \subset D \) (I am using \( \subset \) to mean “proper subset”), but \( A \) and \( C \) are not subsets of \( B \) since \( B \) doesn’t contain either \(-1\) or \( 4 \).

(b) \( A = \{\emptyset, 0, 1\}, B = \{\emptyset, \emptyset\}, C = [0, 1], D = \{\{0, 1\}, \emptyset, \{1\}, \emptyset, \emptyset\} \)

   Solution: The only relationship here is that \( B \subset D \).

(c) \( A = \{x \in \mathbb{N} \mid |x| \leq 4\}, B = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}, C = \{x \in \mathbb{Z} \mid |x| < 5\} \)

   Solution: In this case, \( A \subset B = C \).

2. Considering the definition of interval given in class, explain, as precisely as you can, in a few sentences, why the set \( \{0, 1, 2\} \) is not an interval.

   Solution: Recall that the definition is that \( I \subseteq \mathbb{R} \) is an interval if \( a, b \in I \), and \( a < c < b \), implies that \( c \in I \). Let’s call our subset \( A = \{0, 1, 2\} \). Note that \( 0, 1 \in A \), and \( 0 < \frac{1}{2} < 1 \), but \( \frac{1}{2} \notin A \). Thus, \( A \) is not an interval.

3. Consider the subsets \( \mathbb{Z} \) and \( \mathbb{Q} \) of \( \mathbb{R} \). Based on the definition of interval given in class, are \( \mathbb{Z} \) and \( \mathbb{Q} \) intervals? Explain precisely.

   Solution: We can use the same example as in the previous problem to show that \( \mathbb{Z} \) is not an interval. Since \( 0 < \frac{1}{2} < 1 \) and since \( \frac{1}{2} \notin \mathbb{Z} \), we can see that \( \mathbb{Z} \) is not an interval.

   Now consider the rational numbers 1 and 2. Since \( 1 < \sqrt{2} < 2 \) and since \( \sqrt{2} \notin \mathbb{Q} \), we see that \( \mathbb{Q} \) is also not an interval. (\( \mathbb{R} \) is an interval however.)

4. Calculate \( \mathcal{P}(S) \) for

(a) \( S = \{1, 2, 3\} \)

   Solution: \( \mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \).

(b) \( S = \{a, b, c, d\} \)

   Solution: \( \mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\} \)

(c) \( S = \emptyset \)

   Solution: \( \mathcal{P}(S) = \{\emptyset\} \)

(d) \( S = \{\emptyset\} \)

   Solution: \( \mathcal{P}(S) = \{\emptyset, \{\emptyset\}\} \)

(e) \( S = \{\emptyset, \{\emptyset\}\} \)

   Solution: \( \mathcal{P}(S) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\} \)

1-elt. subsets
5. Let $U = \{1, 2, 3, \ldots, 9, 10\}$, $A = \{1, 7, 9\}$, $B = \{3, 5, 6, 9, 10\}$, and $C = \{1, 2, 4, 8, 9, 10\}$. Calculate:

(a) $(B \cup C)'$
Solution: $B \cup C = \{1, 2, 3, 4, 5, 6, 8, 9, 10\}$, so $(B \cup C)' = \{7\}$.

(b) $B' \cup C'$
Solution: $B' = \{1, 2, 4, 7, 8\}$, $C' = \{3, 5, 6, 7\}$, so $B' \cup C' = \{1, 2, 3, 4, 5, 6, 7, 8\}$

(c) $(A \cap B \cap C)'$
Solution: $A \cap B \cap C = \{9\}$, so $(A \cap B \cap C)' = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$.

(d) $A' \cup B' \cup C'$
Solution: $A' = \{2, 3, 4, 5, 6, 8, 10\}$, $B' = \{1, 2, 4, 7, 8\}$, $C' = \{3, 5, 6, 7\}$, so $A' \cup B' \cup C' = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$.

Note: Since you know the theorem that says $(A \cap B \cap C)' = A' \cup B' \cup C'$, you could use the theorem to tell you that you should get the same answer as in the previous part. Then you wouldn’t need to go through all the work of calculating $A'$, $B'$ and $C'$ as I did here.

(e) $B' \cap C'$
Solution: Again, you could use DeMoivre’s Theorem that says $(B \cup C)' = B' \cap C'$ to say the answer is the same as in part (a), i.e. $\{7\}$.

To do it directly, since $B' = \{1, 2, 4, 7, 8\}$, and $C' = \{3, 5, 6, 7\}$, we have $B' \cap C' = \{7\}$.

(f) $(B \cap C)'$
Solution: This answer is the same as (b), so we have $(B \cap C)' = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
(Note that $B \cap C = \{9, 10\}$.)

(g) $(A \cup C) - (A \cap C)$
Solution: $A \cup C = \{1, 2, 4, 7, 8, 9, 10\}$, $A \cap C = \{1, 9\}$, so $(A \cup C) - (A \cap C) = \{2, 4, 7, 8, 10\}$.

(h) $A \Delta C$
Solution: This is actually the same set as in (g), so we have $A \Delta C = \{2, 4, 7, 8, 10\}$.

(i) $A \Delta (B \Delta C)$
Solution: From parts (g) and (h), we see a general principle which implies that $B \Delta C = (B \cup C) - (B \cap C)$. Hence, since $B \cap C = \{9, 10\}$, we see that $B \Delta C = \{1, 2, 3, 4, 5, 6, 8\}$. Since $A \cap (B \Delta C) = \{1\}$, by the same principle, we have $A \Delta (B \Delta C) = \{2, 3, 4, 5, 6, 7, 8, 9\}$. 

(j) $(A \triangle B) \triangle C$
Solution: Since $A \cap B = \{9\}$, we have $A \triangle B = \{1, 3, 5, 6, 7, 10\}$. Since $(A \triangle B) \cap C = \{1, 10\}$, we see that $(A \triangle B) \triangle C = \{2, 3, 4, 5, 6, 7, 8, 9\}$.

(k) $C - (B - A)$
Since $B \cap A = \{9\}$, we see that $B - A = \{3, 5, 6, 10\}$. Further, since $C \cap (B - A) = \{10\}$, we have $C - (B - A) = \{1, 2, 4, 8, 9\}$.

(l) $(C - B) - A$
Solution: Since $C \cap B = \{9, 10\}$, $C - B = \{1, 2, 4, 8\}$. Further, since $(C - B) \cap A = \{1\}$, we have $(C - B) - A = \{2, 4, 8\}$.

(m) $(C - B) \cap (C - A)$
Solution: From part (l), we have $C - B = \{1, 2, 4, 8\}$. Moreover, $C - A = \{2, 4, 8, 10\}$ (since $C \cap A = \{1, 9\}$). Thus, $(C - B) \cap (C - A) = \{2, 4, 8\}$.

(n) $C - (B \cup A)$
Solution: Now, $B \cup A = \{1, 3, 5, 6, 7, 9, 10\}$ and $C \cap (B \cup A) = \{1, 9, 10\}$, so $C - (B \cup A) = \{2, 4, 8\}$.

6. Let $U = \{1, 2, 3, \ldots, 9, 10\}$. Find specific subsets $A, B, C,$ and/or $X$ of $U$ that contradict (i.e. disprove) the conjectures listed here. For any subsets $A, B, C,$ and $X$ of $U$:

(a) $A - (B - C) = (A - B) - C$
Solution: Looking at parts (k) and (l) of the previous problem, we see that if we take $A = \{1, 2, 4, 8, 9\}$, $B = \{3, 5, 6, 9, 10\}$, and $C = \{1, 7, 9\}$, (in other words, we’re taking the sets from the last problem and switching the names of $A$ and $C$), then $A - (B - C) = \{1, 2, 4, 8, 9\} \neq \{2, 4, 8\} = (A - B) - C$.

(b) $(A - B)' = A - B'$
Solution: Let $A = \{1, 2, 3, 4, 5\}, B = \{1, 3, 5\}$. Then $A - B = \{2, 4\}$ which means $(A - B)' = \{1, 3, 5, 6, 7, 8, 9, 10\}$. On the other hand, $B' = \{2, 4, 6, 7, 8, 9, 10\}$, so $A - B' = \{1, 3, 5\}$. So we see that the two sets $(A - B)'$ and $A - B'$ are not equal.

(c) If $(A \cap C) \subseteq (B \cap C)$, then $A \subseteq B$
Solution: Let $A$ and $B$ be the same sets as in part (b) and let $C = \emptyset$. Then $A \cap C = B \cap C = \emptyset$ and therefore $A \cap C \subseteq B \cap C$ (since $\emptyset \subseteq \emptyset$). But $A \not\subseteq B$ since in fact $B$ is a proper subset of $A$.

(d) If $X \subseteq A$, then $X \cup (A \cap B) = (X \cup A) \cap B$
Solution: Let $X = \{1, 3, 5\}, A = \{1, 2, 3, 4, 5\}, B = \{4, 5, 6, 7\}$. Then $A \cap B = \{4, 5\}$, so $X \cup (A \cap B) = \{1, 3, 4, 5\}$. On the other hand, $X \cup A = A$ since $X \subseteq A$, so $(X \cup A) \cap B = A \cap B = \{4, 5\} \neq \{1, 3, 4, 5\} = X \cup (A \cap B)$.

(e) If $A \times B = A \times C$, then $B = C$
Solution: Let $A = \emptyset, B = \{1, 2\}, C = \{3, 4\}$. Then $A \times B = \emptyset = A \times C$, but $B \neq C$. 

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