MATH 4610 Exercise Set

For the 7th Edition

I.2 # 12, 17, 19, 21, 28
I.3 # 24, 25, 33
I.4 # 19, 30,
I.4 # 21, 29, 32, 34, 35, 37
I.5 # 17, 43, 45, 51, 52, 54, 57
I.6 # 16, 37, 41, 52, 56a
II.8 # 21, 25, 33, 34, 47, 49, 51b
II.9 # 12, 27, 29, 33
II.10 # 28, 29, 30, 33, 36, 39, 43,

Find all abelian groups of order 98000 up to isomorphism.

II.11 # 29, 47, 48, 49

III.13 # 8, 10, 25, 47, 50, 51

Bonus (but required) Fun Questions

1. Find a nontrivial homomorphism from $D_n$ to $Z_2$

2. Let $\langle S, * \rangle$ be the group of all real numbers except $-1$ under the operation $*$ defined by $a * b = a + b + ab$. Show that $\langle S, * \rangle$ is isomorphic to the group $\mathbb{R}^\times$ of nonzero real numbers under multiplication. Actually define an isomorphism $\phi : \mathbb{R}^\times \to S$.

3. Let $G$ be a group. Prove that the permutations $\rho_a : G \to G$, where $\rho_a(x) = xa$ for $a \in G$ and $x \in G$, do form a group $G_1$ (with function composition as the operation) isomorphic to $G$.

4. Let $G$ be a group and let $g$ be one fixed element of $G$. Show that the map $i_g$ defined by $i_g(x) = g^{-1}xg$ for $x \in G$, is an isomorphism of $G$ with itself.

III.14 # 12, 27, 30, 31, 33, 34, 35, 37
III.15 # 37, 40

VII.36 # 15, 17
VII.37 # 4, 8abc, 9, 10
Find $Z(D_n)$ and the commutator subgroup of $D_n$.

Bonus (but required) Fun Questions
1. Show the center of a group of order 60 cannot have order 4.

2. If $H$ is a normal subgroup of a finite group $G$ and $|H| = p^k$ for some prime $p$, show that $H$ is contained in every Sylow-$p$-subgroup of $G$.

3. Let $X$ be a finite $G$-set, where $G$ is a group of order $p^n$ ($p$ a prime) such that $p$ does not divide $|X|$. Show that there exists $x \in X$ such that $x$ is fixed by every element of $G$.

4. Let $G$ be a finite group and $H$ be a subgroup of $G$ such that $|H| = p^k$, where $p$ is a prime and $k$ is a nonnegative integer. Show that

$$[G : H] \equiv [N(H) : H] \pmod{p}.$$ (Hint: Use group actions.)

5. Suppose that $p$ is prime and $|G| = p^n$. If $H$ is a proper subgroup of $G$, prove that $N(H) > H$ (i.e. $H$ is a proper subgroup of $N(H)$). (Hint: use the previous exercise.)

6. Let $G$ be a finite group. Prove that if there exists an element $a \in G$ with exactly two conjugates, then $G$ contains an nontrivial normal subgroup.

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VII.35 # 2, 4, 6
VII.37 # 5, 6
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VII.35 #14, 26 (Don’t just quote theorems from class. We didn’t prove those.)

Bonus (but required) Fun Questions

1. Find the Descending Central Series of $S_4 \times D_8$.

2. Find the Derived Series of $S_4 \times D_8$.

3. Show that the dihedral groups are solvable.

4. Show that a group of order $p^n$ where $p$ is prime is solvable.

5. Give an explicit example of a group $G$ which possesses a normal subgroup $H$ such that both $H$ and $G/H$ are nilpotent, but $G$ is not nilpotent.

6. Show that if $G/Z(G)$ is nilpotent, then $G$ is nilpotent.