§I.4 #21 According to Exercise 12 of Section 2, there are 16 possible binary
operations on a set of 2 elements. How many of these give a structure of a group?
How many of the 19,683 possible binary operations on a set of 3 elements give
a group structure?

Solution: There are only two binary operations on a set of two elements that yield a group.
The rows and columns for the identity are determined by the property of the identity which
determines three of the four available entries in the table. The remaining entry is determined
by the fact that if \( a \neq e \), then \( a^2 \neq a \). Thus, since there are two choices for the identity
element, we have two binary operations that give a group structure.

For sets of three elements, there are only three possibilities. Again the rows and columns
of the multiplication table that contain the identity are determined by the property of the
identity. So there are four entries left.

\[
\begin{array}{c|ccc}
* & e & a & b \\
\hline
 e & e & a & b \\
 a & a & \cdot & \cdot \\
b & b & \cdot & \cdot 
\end{array}
\]

Now \( a^2 \neq a \) since \( a \neq e \) and similarly for \( b \), so \( a^2 = b \) or \( a^2 = e \). If \( a^2 = b \), then \( a^3 = e \) and
similarly \( b^3 = e \) because otherwise either \( a^3 = a \Rightarrow a^2 = e \) or \( a^3 = b = a^2 \Rightarrow a = e \). So
assuming that \( a^2 = b \) determines the entire table. Now assume that \( a^2 = e \). Then by what
we said before, it must also be true that \( b^2 = e \). But \( ab = a \) and \( ab = b \) are both impossible
since then either \( a \) or \( b \) is the identity. So both \( ab \) and \( ba \) must be \( e \). But now \( a^2 = e \) and
\( ab = e \) so \( a \) has two inverses which forces \( a = b \), a contradiction, so \( a^2 = e \) is impossible,
so fixing an identity, we only have one consistent way to fill in the table and get a group.
Hence, since there are three choices for the identity, we get three ways to get a group.

§I.4 #29 Show that if \( G \) is a finite group with identity \( e \) and with an even number
of elements, then there is \( a \neq e \) in \( G \) such that \( a * a = e \).

Solution: Define \( S := \{ g \in G \mid g^{-1} \neq g \} \) and \( T := \{ g \in G \mid g^{-1} = g \} \). Then \( G =
S \cup T \), \( S \cap T = \emptyset \) and so \( |G| = |S| + |T| \). Now \( |S| \) is an even number since every \( g \in G \) is
paired with \( g^{-1} \neq g \) also in \( S \). But since \( |G| \) is even and \( |G| = |S| + |T| \), \( |T| \) must also be
even. Since \( e \in T \), \( T \neq \emptyset \) so \( T \) contains a nonidentity element, say \( a \). But because \( a \in T \),
\( a = a^{-1} \) so \( a * a = e \).
§I.4 #32 Show that every group $G$ with identity $e$ and such that $x \ast x = e$ for all $x \in G$ is abelian. [Hint: Consider $(a \ast b) \ast (a \ast b)$.

Solution: Let $a, b \in G$. Then $a \ast a = e = b \ast b$, and $(a \ast b) \ast (a \ast b) = e$. So, if we right multiply both sides of the last equation by $b \ast a$, we get

$$
(a \ast b) \ast (a \ast b) \ast (b \ast a) = e \ast (b \ast a)
$$

$$
\Rightarrow (a \ast b) \ast (a \ast (b \ast b) \ast a) = b \ast a
$$

$$
\Rightarrow (a \ast b) \ast (a \ast e \ast a) = b \ast a
$$

$$
\Rightarrow (a \ast b) \ast e = b \ast a
$$

$$
\Rightarrow a \ast b = b \ast a.
$$

§I.4 #34 Let $G$ be a group with a finite number of elements. Show that for any $a \in G$, there exists an $n \in \mathbb{Z}^+$ such that $a^n = e$. See Exercise 32 for the meaning of $a^n$. [Hint: Consider $e, a, a^2, a^3, \cdots, a^m$, where $m$ is the number of elements of $G$, and use the cancellation laws.]

Solution: Consider the set $\langle a \rangle^+ = \{a^k \mid k \in \mathbb{Z}^+\}$. It is a subset of $G$ by the closure axiom. But since $G$ is finite, $\langle a \rangle^+$ must be finite so it must be the case that $a^i = a^j$ for some $i \neq j \in \mathbb{Z}^+$. We may assume without loss of generality that $i < j$. So multiply both sides of the equation by $a^{-i}$. Then we have $a^{-i}a^i = a^{-i}a^j \Rightarrow e = a^{j-i}$. Let $n = j - i \in \mathbb{Z}^+$. Then $a^n = e$.

§I.4 #35 Show that if $(a \ast b)^2 = a^2 \ast b^2$ for $a$ and $b$ in a group $G$, then $a \ast b = b \ast a$. See Exercise 32 for the meaning of $a^2$.

Solution: Let $a, b \in G$ and suppose that $(a \ast b)^2 = a^2 \ast b^2$. Then right multiply this equation by $b^{-1}$ and left multiply by $a^{-1}$. So we have

$$
(a^{-1} \ast (a \ast b) \ast (a \ast b) \ast b^{-1}) = (a^{-1} \ast (a \ast a) \ast (b \ast b) \ast b^{-1})
$$

$$
\Rightarrow (a^{-1} \ast a) \ast b \ast a \ast (b \ast b^{-1}) = (a^{-1} \ast a) \ast (a \ast b) \ast (b \ast b^{-1})
$$

$$
e \ast b \ast a \ast e = e \ast a \ast b \ast e
$$

$$
\Rightarrow b \ast a = a \ast b.
$$

§I.4 #37 Let $G$ be a group and suppose that $a \ast b \ast c = e$ for $a, b, c \in G$. Show that $b \ast c \ast a = e$ also.

Solution: We have $a \ast b \ast c = e$. Multiply on the right by $a$ and on the left by $a^{-1}$. Then we get $a^{-1} \ast a \ast b \ast c \ast a = a^{-1} \ast e \ast a \Rightarrow b \ast c \ast a = e$. 

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