§I.5 #17 Let $F$ be the set of all real-valued functions with domain $\mathbb{R}$ and let $\tilde{F}$ be the subset of $F$ consisting of those functions that have a nonzero value at every point in $\mathbb{R}$. Determine whether the given subset of $F$ with the induced operation is (a) a subgroup of $F$ under addition, (b) a subgroup of $\tilde{F}$ under multiplication.

The subset of all $f \in \tilde{F}$ such that $f(0) = 1$.

Solution: Let $A := \{ f \in \tilde{F} \mid f(0) = 1 \}$. Then $A$ is not a subgroup of $F$ under addition since, for example, if $f, g \in A$, then $(f + g)(0) = f(0) + g(0) = 1 + 1 = 2$ so $f + g \notin A$. Hence $A$ is not closed under addition.

The set $A$ is a subgroup of $\tilde{F}$ under multiplication.

Proof. (closure): Let $f, g \in A$ and let $x \in \mathbb{R}$. Then $(fg)(x) = f(x)g(x) \neq 0$ since $f(x), g(x) \neq 0$. Furthermore, since $(fg)(0) = f(0)g(0) = 1 \cdot 1 = 1$, we have $fg \in A$ so $A$ is closed under multiplication.

(identity) Let $e(x) = 1 \forall x \in \mathbb{R}$. Then $e(x) \neq 0 \forall x \in \mathbb{R}$ and $e(0) = 1$, so $e \in A$. Furthermore, $(ef)(x) = 1 \cdot f(x) = f(x) \forall x \in \mathbb{R}$, so $e$ is the identity element of $A$ (and also of $\tilde{F}$).

(inverses) Let $f \in A$. Let $g(x) = \frac{1}{f(x)} \forall x \in \mathbb{R}$. Then, since $f(x) \neq 0 \forall x \in \mathbb{R}, g(x) \neq 0 \forall x \in \mathbb{R}$ and therefore $g \in \tilde{F}$. Also, since $g(0) = \frac{1}{f(0)} = 1, g \in A$. Finally, $(fg)(x) = f(x)g(x) = f(x) \cdot \frac{1}{f(x)} = 1 \forall x \in \mathbb{R}$, so $f^{-1} = g \in A$. 

\[ \square \]

§I.5 #43 Show that if $H$ and $K$ are subgroups of an abelian group $G$, then

\[ \{ hk \mid h \in H \text{ and } k \in K \} \]

is a subgroup of $G$.

Solution: Let $HK := \{ hk \mid h \in H, k \in K \}$.

(closure) Let $h_1k_1, h_2k_2 \in HK$. Then $h_1k_1h_2k_2 = h_1h_2k_1k_2$ since $G$ is abelian. But $h_1h_2 \in H$ and $k_1k_2 \in K$ since $H$ and $K$ are subgroups of $G$ and therefore are closed. Thus, $h_1k_1h_2k_2 \in HK$. 

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(identity) Now \(e\) is an element of both \(H\) and \(K\) since both are subgroups of \(G\), so \(e = e \cdot e \in HK\).

(inverses) Let \(hk \in HK\). Then \((hk)^{-1} = k^{-1}h^{-1} = h^{-1}k^{-1}\) since \(G\) is abelian. But \(h^{-1} \in H\) and \(k^{-1} \in K\) since \(H, K \leq G\), so \((hk)^{-1} \in HK\).

Thus, we have \(HK \leq G\).

\textbf{§I.5 #45} Show that a nonempty subset \(H\) of a group \(G\) is a subgroup of \(G\) if and only if \(ab^{-1} \in H\) for all \(a, b \in H\). (This is one of the more compact criteria referred to prior to Theorem 1.4.14.)

Solution: Suppose \(H\) is such a subset. Now let \(a \in H\). Then \(aa^{-1} = e \in H\) (taking \(b = a\)) so we have \(e \in H\). Next suppose \(a \in H\). Since \(e, a \in H\), we have \(ea^{-1} = a^{-1} \in H\) so we have inverses. Finally, if \(a, b \in H\), then \(b^{-1} \in H\) by what we have just proved. Thus, \(a, b^{-1} \in H\) so \(a(b^{-1})^{-1} = ab \in H\) and we have closure. Hence, \(H\) is a subgroup of \(G\).

\textbf{§I.5 #51} Let \(G\) be a group and let \(a\) be one fixed element of \(G\). Show that

\[
H_a = \{ x \in G \mid xa = ax \}
\]

is a subgroup of \(G\).

Solution: Let \(x, y \in H_a\). Then \((xy)a = x(ya) = x(ay) = (xa)y = (ax)y = a(xy)\), so \(xy \in H_a\) so we have closure. Also, \(ea = ae = a\), so \(e \in H_a\). Finally, suppose \(x \in H_a\). Then \(xa = ax \Rightarrow x^{-1}(xa)x^{-1} = x^{-1}(ax)x^{-1} \Rightarrow (x^{-1}x)(ax^{-1}) = (x^{-1}a)(xx^{-1}) \Rightarrow ax^{-1} = x^{-1}a\). So \(x^{-1} \in H_a\).

Another way to say the last part (about inverses) is to say that

\[
x^{-1}a = x^{-1}a(xx^{-1}) = x^{-1}(ax)x^{-1} = x^{-1}(xa)x^{-1} = (x^{-1}x)ax^{-1} = ax^{-1}.
\]

\textbf{§I.5 #52} Generalizing Exercise 51, let \(S\) be any subset of a group \(G\).

\textbf{a. Show that} \(H_S = \{ x \in G \mid xs = sx \text{ for all } s \in S \}\) \textbf{is a subgroup of} \(G\).

Solution: Let \(x, y \in H_S\) and let \(s \in S\). Then \((xy)s = x(yx) = x(sy) = (xs)y = s(xy)\), so \(xy \in H_S\) so we have closure. Also, \(es = se = s\), so \(e \in H_S\). Finally, suppose \(x \in H_S\). Then \(xs = sx \Rightarrow x^{-1}(xs)x^{-1} = x^{-1}(sx)x^{-1} \Rightarrow (x^{-1}x)(sx^{-1}) = (x^{-1}s)(xx^{-1}) \Rightarrow sx^{-1} = x^{-1}s\). So \(x^{-1} \in H_S\).

\textbf{b. In reference to part (a), the subgroup} \(H_G\) \textbf{is the center of} \(G\). \textbf{Show that} \(H_G\) \textbf{is an abelian group}.

Solution: By part (a), by taking \(S = G\), \(H_G\) is a subgroup of \(G\). Now let \(x, y \in H_G\). Then since \(x \in H_G, y \in G, xy = yx\) and so \(H_G\) is abelian.

\textbf{§I.5 #53} Let \(H\) be a subgroup of a group \(G\). For \(a, b \in G\), let \(a \sim b\) if and only if \(ab^{-1} \in H\). Show that \(\sim\) is an equivalence relation on \(G\).

Solution: \((\sim\) is Reflexive) Since \(H\) is a subgroup, \(aa^{-1} = e \in H\) so \(a \sim a\).

\((\sim\) is Symmetric) Let \(a, b \in G\). If \(a \sim b\), then \(ab^{-1} \in H\). But then since \(H \leq G, (ab^{-1})^{-1} = (b^{-1})^{-1}a^{-1} = ba^{-1} \in H\), so \(b \sim a\).
(∼ is Transitive) Let a, b, c ∈ G and suppose a ∼ b and b ∼ c. Then \(ab^{-1} ∈ H\) and \(bc^{-1} ∈ H\). Since \(H ≤ G\), \((ab^{-1})(bc^{-1}) = ac^{-1} ∈ H\), so \(a ∼ c\).
Thus, ∼ is an equivalence relation.

§I.5 #54 For sets \(H\) and \(K\), we define the intersection \(H ∩ K\) by
\[
H ∩ K = \{x \mid x ∈ H \text{ and } x ∈ K\}.
\]
Show that if \(H ≤ G\) and \(K ≤ G\), then \(H ∩ K ≤ G\).

Solution: (closure) Suppose \(a, b ∈ H ∩ K\). Then \(a, b ∈ H\) which implies \(ab ∈ H\), and \(a, b ∈ K\) which implies \(ab ∈ K\). Thus, \(ab ∈ H ∩ K\).

(identity) Since \(e ∈ H\) and \(e ∈ K\), \(e ∈ H ∩ K\).

(inverses) Suppose \(a ∈ H ∩ K\). Then \(a^{-1} ∈ H\) and \(a^{-1} ∈ K\) since \(H, K ≤ G\). Thus, \(a^{-1} ∈ H ∩ K\).
We have now shown that \(H ∩ K ≤ G\).

§I.5 # 57 Show that a group with no proper nontrivial subgroups is cyclic.

Solution: Suppose \(a ≠ e, a ∈ G\). Then \(⟨a⟩ ≤ G\). But if \(G\) has no proper nontrivial subgroups, and \(a ≠ e\), it must be the case that \(⟨a⟩ = G\). But then \(G\) is cyclic.